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Quadratic Splines and Histospline Projections

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1. INTRODUCTION AND NOTATIONS

Let Δ_n $(n > 1)$ denote an arbitrary but fixed partition of the unit interval $I = [0, 1]$, i.e., $\Delta_n : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. By $\mathrm{Sp}(2, \Delta_n)$ we denote the space of quadratic spline functions determined by the above partition Δ_n. Namely, $s \in \mathrm{Sp}(2, \Delta_n)$ if and only if the following conditions are satisfied:

(i) In each subinterval $[x_{i-1}, x_i]$ $(i = 1, 2,..., n)$, s is an algebraic polynomial of degree 2 or less;

(ii) $s \in C^1(I)$.

It is a well known fact that $\mathrm{Sp}(2, \Delta_n)$ is a linear subspace of $C(I)$, and that $\dim \mathrm{Sp}(2, \Delta_n) = n + 2$. Let L_n^2 denote a projection (linear, bounded and idempotent map) with domain $C(I)$ and range $\mathrm{Sp}(2, \Delta_n)$. If $\|\cdot\|_C$ stands for the sup-norm on the interval I, then the operator norm is given by a familiar formula

$$\| L_n^2 \| = \sup_{\|f\|_C \leqslant 1} \| L_n^2 f \|_C \qquad (f \in C(I)). \tag{1.1}$$

In this note, some examples of the operators L_n^2 are given, as well as some results concerning their norms. The size of the norm L_n^2 is important, for

$$\|f - L_n^2 f\|_C \leqslant (1 + \| L_n^2 \|) \operatorname{dist}(f, \mathrm{Sp}(2, \Delta_n))$$

(for the most general form of this inequality, see, e.g., [5]).

Let $\{L_n^2\}$ denote some class of projections from $C(I)$ onto $\mathrm{Sp}(2, \Delta_n)$. Then $\bar{L}_n^2 \in \{L_n^2\}$ is minimal if $\| \bar{L}_n^2 \| \leqslant \| L_n^2 \|$ for all L_n^2. For some recent results concerning minimal projections, see [9]. Bounds for the norm of certain spline projections are given in [6–8] and [12].

In Section 2 we introduce the projections L_n^2 determined by the conditions

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$(L_n^2 f)(x_i) = f(x_i)$ $(i = 0, 1, \ldots, n)$ and $(L_n^2 f)'(x_0) = 0$. The norm of L_n^2 and its upper bound for a minimal projection $\bar{L}_n^2$ are given in Theorems 2.1 and 2.2, respectively. The next section is devoted to the so-called quadratic histospline projections (denoted by P_n^2, Q_n^2, R_n^2), determined by the conditions

$$\int_{x_{i-1}}^{x_i} [(P_n^2 f)(x) - f(x)]\, dx = 0 \qquad (i = 1, 2, \ldots, n; f \in C(I)), \tag{1.2}$$

with appropriate boundary conditions for the spline $P_n^2 f$ $(Q_n^2 f, R_n^2 f)$. Upper bounds for the norm of P_n^2 (Q_n^2, R_n^2) are given. Splines satisfying (1.2) were introduced in [2]. Schoenberg [14] called these splines histosplines. For further results concerning histosplines see, e.g., [1, 3, 11, 13–16].

2. Quadratic Spline Projections

For simplicity of further notations let $h_i = x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$, $y_i = s(x_i)$, $m_i = s'(x_i)$ $(i = 0, 1, \ldots, n)$ where $s \in \mathrm{Sp}(2, \Delta_n)$. If $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, then the spline function s may be written in terms y_i and m_i in the following way

$$s(x) = y_{i-1} + m_{i-1}(x - x_{i-1}) + \frac{m_i - m_{i-1}}{2h_i}(x - x_{i-1})^2 \qquad (i = 1, 2, \ldots, n). \tag{2.1}$$

Now we assume that the real numbers $y_i (i = 0, 1, \ldots, n)$ and m_0 $(= s'(x_0))$ are given. From conditions $s \in C(I)$ and with the help of (2.1), we obtain

$$m_i + m_{i-1} = 2(y_i - y_{i-1})/h_i \equiv 2d_i \qquad (i = 1, 2, \ldots, n). \tag{2.2}$$

Solving this difference equation we have a simple formula for determining first derivatives m_i at knots, namely

$$m_i = 2 \sum_{l=1}^{i} (-1)^{i+l} d_l + (-1)^i m_0 \qquad (i = 1, \ldots, n). \tag{2.3}$$

Thus from (2.1) and (2.3) we have the following

Lemma 2.1. *For $n > 1$, arbitrary but fixed partition Δ_n, and for arbitrary real numbers y_i $(i = 0, 1, \ldots, n)$ and m_0, there exists exactly one spline function $s \in \mathrm{Sp}(2, \Delta_n)$ satisfying the following conditions $s(x_i) = y_i$ $(i = 0, 1, \ldots, n)$ and $s'(x_0) = m_0$.*

In [9], the authors considered an interpolating scheme as in Lemma 2.1 but with periodic boundary conditions.

For our further use, we introduce the so-called fundamental spline functions $s_j \in \mathrm{Sp}(2, \Delta n)$ $(j = 0, 1, \ldots, n)$ satisfying the conditions

$$s_j(x_i) = \delta_{ij} \qquad (i, j = 0, 1, \ldots, n). \tag{2.4}$$

Also assume that for all $j = 0, 1, \ldots, n$

$$s_j'(x_0) = 0. \tag{2.5}$$

From Lemma 2.1, it follows that s_j always exists for $j = 0, 1, \ldots, n$. In the next lemma an explicit formula for s_j is given.

LEMMA 2.2. *For an arbitrary partition Δ_n, the fundamental spline functions s_j $(j = 0, 1, \ldots, n)$ are given by the formula*

$$\begin{aligned}
s_j(x) &= 0, && x_0 \leqslant x \leqslant x_{j-1}, \\
&= \left(\frac{x - x_{j-1}}{h_j}\right)^2, && x_{j-1} \leqslant x \leqslant x_j, \\
&= 1 + \frac{2}{h_j h_{j+1}}(x_{j+1} - x)(x - x_j) - \left(\frac{x - x_j}{h_{j+1}}\right)^2, && x_j \leqslant x \leqslant x_{j+1}, \\
&= (-1)^{l+1} \frac{m_{j+1}}{h_{j+l+1}}(x - x_{j+l})(x_{j+l+1} - x), && x_{j+l} \leqslant x \leqslant x_{j+l+1}
\end{aligned} \tag{2.6}$$

where $m_{j+1} = -2(1/h_j + 1/h_{j+1})$ $(j = 0, 1, \ldots, n-1;\ 1/h_0 = 0)$.

Proof. Let for the fixed value of the index j $(j = 0, 1, \ldots, n)$ $m_i \equiv s_j'(x_i)$. First, the numbers m_i will be calculated.

Case 1. $j = 0$. From (2.2) conditions (2.4) and (2.5) we have $m_0 = 0$, $m_1 = -2/h_1$, $m_l = (-1)^{l+1} m_1$ $(l = 2, 3, \ldots, n)$.

Case 2. $j = n$. In a manner similar to the above, we obtain $m_l = 0$ for $l = 0, 1, \ldots, n-1$ and $m_n = 2/h_n$.

Case 3. $j = 1, 2, \ldots, n-1$. By virtue of (2.4), (2.5) and (2.2), we obtain $m_i = 0$ for $i = 0, 1, \ldots, j-1$, and further $m_j = 2/h_j$, $m_{j+1} = -2(1/h_j + 1/h_{j+1})$, $m_{j+l} = (-1)^{l+1} m_{j+1}$ $(l \geqslant 1; j + l \leqslant n)$. Then (2.6) is an obvious consequence of the Hermite interpolation formula.

COROLLARY 2.1. *Under the assumptions of Lemma* 2.2, *we have*

$$\begin{aligned}
\operatorname{sgn} s_j(x) &= 0, && x_0 \leqslant x \leqslant x_{j-1}, \\
&= 1, && x_{j-1} < x < x_{j+1}, \\
&= (-1)^l, && x_{j+l} < x < x_{j+l+1} \quad (l \geqslant 1; j + l \leqslant n - 1).
\end{aligned} \tag{2.7}$$

Let $f \in C(I)$. We define the projection L_n^2 in the following way

$$(L_n^2 f)(x) = \sum_{j=0}^{n} f(x_j)\, s_j(x), \tag{2.8}$$

where $s_j \in \mathrm{Sp}(2, \Delta_n)$. It is obvious by virtue of (2.4) and (2.5) that $(L_n^2 f)(x_i) = f(x_i)$ $(i = 0, 1,..., n)$ and $(L_n^2 f)'(x_0) = 0$. If

$$\Lambda_n^2(x) = \sum_{j=0}^{n} | s_j(x)| \qquad (x \in I)$$

denotes the so-called Lebesgue function connected with the projection L_n^2, then

$$\| L_n^2 \| = \| \Lambda_n^2 \|_C . \tag{2.9}$$

Now we are able to prove the following

THEOREM 2.1. *If L_n^2 is defined by* (2.8), *then*

$$\| L_n^2 \| = 1 + \max_{1 \leqslant i \leqslant n} h_i \sum_{j=1}^{i-1} 1/h_j . \tag{2.10}$$

Proof. Let $x \in [x_{i-1}, x_i]$ $(i = 1, 2,..., n)$. Then by virtue of (2.6) and (2.7) we obtain

$$\Lambda_n^2(x) = \sum_{j=0}^{n} | s_j(x)| = \sum_{j=0}^{i} | s_j(x)| = \sum_{j=0}^{i-2} | s_j(x)| + s_{i-1}(x) + s_i(x).$$

Hence

$$\Lambda_n^2(x) = 1 + \frac{2}{h_{i-1}h_i}(x_i - x)(x - x_{i-1}) + \sum_{j=0}^{i-2} | s_j(x)| \qquad (x \in [x_{i-1}, x_i]). \tag{2.11}$$

For simplicity of further notation, let $w_i(x) = (x_i - x)(x - x_{i-1})$. From Lemma 2.2 we have $| s_0(x)| = (2/h_i h_1)\, w_i(x)$, $i = 2, 3,..., n + 1$, $| s_j(x)| = 2/h_i(1/h_j + 1/h_{j+1})\, w_i(x)$ $(j = 1, 2,..., i - 2)$; $i \geqslant j + 2$. Hence (2.11) becomes

$$\Lambda_n^2(x) = 1 + \frac{2}{h_i}\left[\frac{1}{h_{i-1}} + \frac{1}{h_1} + \sum_{j=1}^{i-2}\left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right)\right] w_i(x)$$

$$= 1 + \frac{4}{h_i}\sum_{j=1}^{i-1}\frac{1}{h_j}\, w_i(x).$$

Further $\max_{x_{i-1} \leqslant x \leqslant x_i} \Lambda_n^2(x) = 1 + h_i \sum_{j=1}^{i-1} 1/h_j$ $(i = 1, 2,..., n)$, and (2.9) implies the formula (2.10). ∎

Let Δ_n and $\tilde{\Delta}_n$ be two different partitions of the unit interval I such that the mesh sizes h_i and $\tilde{h}_i$ $(i = 1, 2,..., n)$ fulfill the inequalities

$$0 < h_n \leqslant h_{n-1} \leqslant \cdots \leqslant h_2 \leqslant h_1 < 1, \tag{2.12}$$

$$0 < \tilde{h}_1 \leqslant \tilde{h}_2 \leqslant \cdots \leqslant \tilde{h}_{n-1} \leqslant \tilde{h}_n < 1, \tag{2.13}$$

where $\sum_{i=1}^{n} h_i = \sum_{i=1}^{n} \tilde{h}_i = 1$. Further let $e_i = h_i \sum_{j=1}^{i-1} 1/h_j$, and $\tilde{e}_i = \tilde{h}_i \sum_{j=1}^{i-1} 1/\tilde{h}_j$ $(i = 2, 3,..., n)$.

It follows from the proof of Theorem 2.1 that the Lebesgue function $\Lambda_n^2(x)$ is strictly concave in each subinterval $[x_{i-1}, x_i]$ $(i = 2, 3,..., n)$ and $\Lambda_n^2(x) \equiv 1$ for $x \in [x_0, x_1]$. Thus the numbers $1 + e_i$ and $1 + \tilde{e}_i$ are local maxima of the functions $\Lambda_n^2(x)$ and $\tilde{\Lambda}_n^2(x)$, respectively. We can prove the following

COROLLARY 2.2. *If the partitions Δ_n and $\tilde{\Delta}_n$ are such that* (2.12) *and* (2.13) *hold, then $e_i \leqslant \tilde{e}_i$ $(i = 2, 3,..., n)$. Hence $\| L_n^2 \| \leqslant \| \tilde{L}_n^2 \|$.*

Proof. From (2.12) and the definition of e_i we obtain

$$e_i \leqslant (i - 1)\, h_i/h_{i-1} \qquad (i = 2, 3,..., n). \tag{2.14}$$

Similarly (2.13) implies

$$(i - 1)\, \tilde{h}_i/\tilde{h}_{i-1} \leqslant \tilde{e}_i \qquad (i = 2, 3,..., n). \tag{2.15}$$

From our assumptions, $h_i/h_{i-1} \leqslant 1$ and $\tilde{h}_i/\tilde{h}_{i-1} \geqslant 1$ $(i = 2, 3,..., n)$. Hence and from (2.14) and (2.15) one obtains

$$e_i \leqslant (i - 1) \leqslant (i - 1)\, \tilde{h}_i/\tilde{h}_{i-1} \leqslant \tilde{e}_i . \quad \blacksquare$$

From Corollary 2.2 it follows that the minimal projection $\bar{L}_n^2$ is not contained in some subclass of projections (of the form (2.8)) determined by the partitions like (2.13). Now we give an upper bound for the norm of the minimal projection $\bar{L}_n^2$.

THEOREM 2.2. *If $\bar{L}_n^2$ is a minimal projection among all projections of the form* (2.8), *then for every $n > 1$, the following upper bound is valid*

$$\| \bar{L}_n^2 \| \leqslant 3 - 2^{2-n} \leqslant 3.$$

Proof. Let Δ_n be defined as follows, $h_i = 2^{-i}$ $(i = 1, 2,..., n - 1)$ and $h_n' = 2^{1-n}$ (the condition $\sum_{i=1}^{n} h_i = 1$ is satisfied). We have

$$h_i \sum_{j=1}^{i-1} \frac{1}{h_j} = 1 - 2^{1-i}, \qquad i = 1, 2,..., n - 1,$$
$$= 2 - 2^{2-n}, \qquad i = n.$$

Hence by virtue of (2.10), $\| L_n^2 \| \leqslant 3 - 2^{2-n} \leqslant 3$ and further the definition of a minimal projection gives the desired result. ∎

Meinardus and Taylor [9] proved that a minimal projection in the class considered in their paper has a norm $(n + 1)/2$ (n-odd). From Theorem 2.2 it follows that a minimal projection in the class defined by (2.8) has a smaller norm even for small values of n.

3. Quadratic Histospline Projections

Now we introduce some additional notation. Let $h = \max_{1 \leqslant i \leqslant n} h_i$, $\underline{h} = \min_{1 \leqslant i \leqslant n} h_i$, $K_n = h/\underline{h}$ denote a global mesh ratio for partition Δ_n, $a_i = h_{i+1}/(h_i + h_{i+1})$, $c_i = 1 - a_i$ $(i = 1, 2, ..., n - 1)$, $F_i = \int_{x_{i-1}}^{x_i} f(x)\, dx$ for given $f \in C(I)$. In this section, upper bounds for the norms of the projections P_n^2, Q_n^2 and R_n^2(defined below) are given. As in the previous section, the projections P_n^2, Q_n^2 and R_n^2 are defined on $C(I)$ with values in $\mathrm{Sp}(2, \Delta_n)$. Let $P_n^2 f \equiv s \in \mathrm{Sp}(2, \Delta_n)$ be determined by the conditions

$$\int_{x_{i-1}}^{x_i} [(P_n^2 f)(x) - f(x)]\, dx = 0 \qquad (i = 1, 2, ..., n). \tag{3.1}$$

Additionally for the histospline $P_n^2 f$ the following boundary conditions are imposed

$$(P_n^2 f)(x_0) = f(x_0), \qquad (P_n^2 f)(x_n) = f(x_n). \tag{3.2}$$

Respectively for $Q_n^2 f$ the following boundary conditions are assumed

$$(Q_n^2 f)'(x_0) = (Q_n^2 f)'(x_n) = 0, \tag{3.3}$$

where $Q_n^2 f \equiv s \in \mathrm{Sp}(2, \Delta_n)$ satisfy conditions as in (3.1). Similarly we define the projection R_n^2 with the additional assumption that $f(0) = f(1)$. In this case, the boundary conditions for $R_n^2 f$ are periodic, i.e.,

$$(R_n^2 f)^{(j)}(x_0) = (R_n^2 f)^{(j)}(x_n) \qquad (j = 0, 1). \tag{3.4}$$

Let $x \in [x_{i-1}, x_i]$, $t = (x - x_{i-1})/h_i$. The following formula for $(P_n^2 f)(x)$ valid (see, e.g., [11, 13, 15])

$$(P_n^2 f)(x) = f(x_{i-1})(1 - t)(1 - 3t) + f(x_i)t, (3t - 2t) + 6t(1 - t)\, F_i/h_i, \tag{3.5}$$

where the numbers $y_i = f(x_i)$ $(i = 0, 1, ..., n)$ are the solution of the following system of linear equations (see, e.g., [11, 13, 15])

$$a_i y_{i-1} + 2y_i + c_i y_{i+1} = 3(a_i F_i/h_i + c_i F_{i+1}/h_{i+1}) \qquad (i = 1, 2, ..., n - 1) \tag{3.6}$$

(by virtue of (3.2) the numbers y_0 and y_n are given). Similarly for the histospline $Q_n^2 f$ we have for $x \in [x_{i-1}, x_i]$

$$(Q_n^2 f)(x) = F_i/h_i + h_i[(3t^2 - 1)\, m_i - (3t^2 - 6t + 2)\, m_{i-1}]/6, \quad (3.7)$$

and the appropriate system of linear equations with unknowns m_i is the following:

$$c_i m_{i-1} + 2m_i + a_i m_{i+1} = \frac{6}{h_i + h_{i+1}} (F_{i+1}/h_{i+1} - F_i/h_i) \quad (i = 1, 2, \ldots, n-1) \quad (3.8)$$

(by virtue of (3.3) $m_0 = m_n = 0$). For the histospline $R_n^2 f$, formula (3.5) is applicable.

Now we are able to prove the following

THEOREM 3.1. *For the projections P_n^2, Q_n^2 and R_n^2 the following estimates hold*

$$\| P_n^2 \| \leqslant 4\tfrac{1}{2}, \qquad \| Q_n^2 \| \leqslant 1 + 3K_n, \qquad \| R_n^2 \| \leqslant 4\tfrac{1}{2}. \quad (3.9)$$

Proof. We only sketch the proof because it is quite similar to the proof of Theorem 3.1 in [7]. Using a diagonal dominance argument to the systems (3.6) and (3.8) one has

$$\max_{0 \leqslant j \leqslant n} | y_j | \leqslant 3 \| f \|_C \quad \text{and} \quad \max_{0 \leqslant j \leqslant n} | m_j | \leqslant 6 \| f \|_C / \underline{h}.$$

Hence and from (3.5) and (3.7) we obtain the first and second inequality in (3.9). Quite similarly one can prove that the last inequality holds. ∎

Note added in proof. It can be proved that the following uniform upper bound for the norms of P_n^2, Q_n^2 and R_n^2 holds: $\| P_n^2 \|$, $\| Q_n^2 \|$, $\| R_n^2 \| \leqslant 3$ for all $n > 1$ and arbitrary partition Δ_n.

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